MTH 507 Midterm Solutions

1. Let X be a path connected, locally path connected, and semilocally simply connected space. Let H_0 and H_1 be subgroups of $\pi_1(X, x_0)$ (for some $x_0 \in X$) such that $H_0 \leq H_1$. Let $p_i : X_{H_i} \to X$ (for $i = 0, 1$) be covering spaces corresponding to the subgroups H_i. Prove that there is a covering space $f : X_{H_0} \to X_{H_1}$ such that $p_1 \circ f = p_0$.

   Solution. Choose $\widetilde{x_0}, \widetilde{x_1} \in p^{-1}(x_0)$ so that $p_i : (X_{H_i}, \widetilde{x_i}) \to (X, x_0)$ and $p_{i*}(X_{H_i}, \widetilde{x_i}) = H_i$ for $i = 0, 1$. Since $H_0 \leq H_1$, by the Lifting Criterion, there exists a lift $f : X_{H_0} \to X_{H_1}$ of p_0 such that $p_1 \circ f = p_0$. It remains to show that $f : X_{H_0} \to X_{H_1}$ is a covering space.

   Let $y \in X_{H_1}$, and let U be a path-connected neighbourhood of $x = p_1(y)$ that is evenly covered by both p_0 and p_1. Let $V \subset X_{H_1}$ be the slice of $p_1^{-1}(U)$ that contains y. Denote the slices of $p_0^{-1}(U)$ by $\{V'_z : z \in p_0^{-1}(x)\}$. Let C denote the subcollection $\{V'_z : z \in f^{-1}(y)\}$ of $p_0^{-1}(U)$. Every slice V_z is mapped by f into a single slice of $p_1^{-1}(U)$, as these are path-connected. Also, since $f(z) \in p_1^{-1}(y)$, $f(V'_z) \subset V$ iff $f(z) = y$. Hence $f^{-1}(V)$ is the union of the slices in C.

   Finally, we have that for $V'_z \in C$, $(p_1|_V)^{-1} \circ (p_0|_{V'_z}) = f|_{V'_z}$. Hence $f|_{V'_z}$ is a homeomorphism, and V is a evenly covered neighborhood of y.

2. A *homomorphism* between two covering spaces $p_1 : \widetilde{X_1} \to X$ and $p_2 : \widetilde{X_2} \to X$ is a map $f : \widetilde{X_1} \to \widetilde{X_2}$ so that $p_1 = p_2 \circ f$.

   (a) Classify all the covering spaces of S^1 up to isomorphism.

   (b) Find all homomorphisms between these covering spaces.

   Solution. (a) Let $x_0 = (1, 0)$, then by Classification of Covering Spaces, the basepoint-preserving isomorphism classes of covering spaces of (S^1, x_0) correspond to the subgroups of $\pi_1(S^1, x_0) \cong \mathbb{Z}$. Each non-trivial subgroup $m\mathbb{Z} \leq \mathbb{Z}$ corresponds to the covering space $p_m : (S^1, \widetilde{x_0}) \to (S^1, x_0)$, where $p_m(z) = z^m$ and $\widetilde{x_0}$ is an m^{th} root of unity in $S^1 \subset \mathbb{C}$. The trivial subgroup corresponds to the universal cover $p : \mathbb{R} \to S^1$ given by $p(s) = e^{i(2\pi s)}$.

   (b) Let $p_m : (S^1, \widetilde{x_0}) \to (S^1, x_0)$ and $p_n : (S^1, \widetilde{x_1}) \to (S^1, x_0)$ be covering spaces. Any homomorphism $f : (S^1, \widetilde{x_0}) \to (S^1, \widetilde{x_1})$ must induce a homomorphism $f_* : \mathbb{Z}_m \to \mathbb{Z}_n$. Such a homomorphism exists iff $m \mid n$, and when $m \mid n$, f_* is the natural injection $\mathbb{Z}_m \hookrightarrow \mathbb{Z}_n$. There can both be no homomorphism between universal cover $p : \mathbb{R} \to S^1$ to any of the finite-sheeted covering spaces mentioned above, as its fundamental group is trivial. Finally any homomorphism f from the universal cover to itself should be an isomorphism (by the Lifting Criterion) which satisfies $p \circ f = p$. Since $p(s) = e^{i(2\pi s)}$, f has to be of the form $s \mapsto s + k$, where k is a integer.

3. Let (X, x_0) and (Y, y_0) be topological spaces.

   (a) Show that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. [Hint: Use the projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$.]

   (b) Compute the fundamental group of the solid torus.

   Solution. (a) Please see Theorem 60.1 (Page 371) in Munkres.

   (b) The solid torus $X \approx D^2 \times S^1$. Hence $\pi_1(X) \approx \pi_1(D^2) \times \pi_1(S^1)$, from part(a). Since D^2 is simply connected, we have that $\pi_1(X) \cong \mathbb{Z}$.

4. Find all 2-sheeted covering spaces of the torus $S^1 \times S^1$ up to isomorphism.

   Solution. By the Classification of Covering Spaces, the isomorphism class of any 2-sheeted covering space will correspond to a subgroup to $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ of index 2. The two obvious nontrivial subgroups are $2\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times 2\mathbb{Z}$, which correspond to the coverings $z^2 \times w : S^1 \times S^1 \to S^1 \times S^1$ and $z \times w^2 : S^1 \times S^1 \to S^1 \times S^1$ respectively.

   There is only one other nontrivial subgroup. Before we describe this subgroup, note that any such subgroup will be isomorphic to the kernel of a homomorphism $\mathbb{Z}^2 \to \mathbb{Z}_2$. The third nontrivial subgroup is isomorphic to the kernel of the homomorphism that maps both of the standard generators of $\mathbb{Z}^2$ to $1 \in \mathbb{Z}_2$. Explicitly, this subgroup can be described as $\{(x, y) \in \mathbb{Z}^2 \mid x + y \pmod 2 = 0\}$. Can you describe the covering space this subgroup corresponds to?

5. Consider the real projective n-space $\mathbb{R}P^n$ obtained by identifying each point $x \in S^n$ with its antipode $-x$.

   (a) Compute its fundamental group.

   (b) Find all its covering spaces up to isomorphism.

(c) Show that every map from $\mathbb{R}P^2 \to S^1$ is nullhomotopic.

Solution. (a) We know from class that the quotient map $q : S^n \to \mathbb{R}P^n$ for ($n \geq 2$) is a 2-fold universal covering space. (Note that we are assuming here that S^n is simply connected.) Therefore, the lifting correspondence is bijective, and consequently, $\pi_1(\mathbb{R}P^n)$ is a group that has exactly 2 elements. Therefore, $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$.

(b) Up to isomorphism, any covering space of $\mathbb{R}P^n$ will correspond to a subgroup to $\mathbb{Z}_2$. As the only subgroups of $\mathbb{Z}_2$ are itself and the trivial group, they will correspond to the covering spaces of $id : \mathbb{R}P^2 \to \mathbb{R}P^2$ and $q : S^n \to \mathbb{R}P^n$ (which in this case is the universal cover) respectively.

(c) Suppose that $f : \mathbb{R}P^2 \to S^1$ is a continuous map. Then it induces a homomorphism $h_* : \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 \to \pi_1(S^1) \cong \mathbb{Z}$, which has be trivial. By the Lifting Criterion, f lifts to a map $\widetilde{f} : \mathbb{R}P^2 \to \mathbb{R}$ such that $p \circ \widetilde{f} = f$, where $p : \mathbb{R} \to S^1$ is the standard universal covering. Since $\mathbb{R}$ is contractible, f has be homotopic (via some H) to a constant map. Hence $p \circ H$ is a homotopy from f to a constant map.

6. Let $r : S^1 \to S^1$ be a reflection of the circle (e.g. $(x, y) \to (-x, y)$ in the plane). The *Klein bottle* K is the quotient space of $[0, 1] \times S^1$ under the following equivalence relation: $(0, z) \sim (1, r(z))$ for all $z \in S^1$, and (t, z) is not equivalent to anything except itself, for $t = 0, 1$.

   (a) Explain why K is compact.

   (b) Let $C_1 \subset K$ be (the image of) the circle $1 \times S^1$, and let $C_2 \subset K$ be a small embedded circle inside $(\frac{1}{2}, \frac{3}{2}) \times S^1$. There is a continuous map $g : K \to \mathbb{R}^3$ such that $g|_{K-C_1}$ and $g|_{K-C_2}$ are injective. Assuming g exists as described, use Urysohn's Lemma to construct a continuous map of K into $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, which is an imbedding.

Solution. (a) Since the quotient map is a continuous map, and $[0, 1] \times S^1$ is compact, it follows that K has to be compact.

(b) Since K is Hausdorff and $C_1, C_2 \approx S^1$, they have to be closed sets in K. As K is also compact, it is a normal space. By Urysohn's Lemma, we obtain a function $f : K \to [0, 1]$ such that $f(C_1) = \{0\}$ and $f(C_2) = \{1\}$. Composing f with the inclusion $[0, 1] \hookrightarrow \mathbb{R}$, we obtain a function $f' : K \to \mathbb{R}$ such that $f'(C_1) = \{0\}$ and $f'(C_2) = \{1\}$.

Now define $h : K \to \mathbb{R}^3 \times \mathbb{R}$ by $h(x) = (g(x), f'(x))$. This is map is continuous, as it has continuous coordinate functions. Consider two distinct points $x, y \in K$. If both x and y are in C_1 and C_2 then $f'(x) \neq f'(y)$, and so $h(x) \neq h(y)$. Without loss of generality, if we assume that C_1 does not contain x or y, then $x, y \in K \setminus C_1$, and g is injective on this subset. Hence $g(x) \neq g(y)$, and so $h(x) \neq h(y)$.

Finally, h is a continuous and injective map from a compact space to a Hausdorff space. This implies that h has to be an imbedding.

7. Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps.

   (a) If $h \simeq k(via\, H)$ such that $H(x_0, t) = y_0$ for all t, then show that $h_* = k_*$.

   (b) Using (a) show that the inclusion $j : S^n \to R^{n+1} \setminus \{0\}$ induces an isomorphism of fundamental groups. [Hint: Use the natural retraction map $r : R^{n+1} \setminus \{0\} \to S^n$].

   Solution. Please see Lemma 58.2 and Theorem 58.2 (Page 360) from Munkres.

8. **[Bonus]** Let $f : (S^1, x_0) \to (S^1, x_1)$ be a continuous map. Then the induced homomorphism $f_* : \pi_1(S^1, x_0)(\cong \mathbb{Z}) \to \pi_1(S^1, x_1)(\cong \mathbb{Z})$ is completely determined by the integer d given by $f_*([\alpha_0]) = d[\alpha_1]$, where $[\alpha_i]$ is a generator $\pi_1(S^1, x_i)$ (for $i = 0, 1$) that represents $1 \in \mathbb{Z}$. This integer d is called the *degree* of f (denoted by $deg(f)$).

   (a) Show that if $f \simeq g$, then $deg(f) = deg(g)$.

   (b) Show that if f is a homeomorphism, then $deg(f) = \pm 1$. In particular, show that the $deg(a) = -1$, when a is the antipodal map.

   Solution. (a) Note that $deg(f)$ is independent of basepoint, for if we choose a different basepoint $y_0 \in X$ with $f(y_0) = y_1$, then there exists isomorphisms $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $\hat{\beta} : \pi_1(X, x_1) \to \pi_1(Y, y_1)$ such that $f_* \circ \hat{\alpha} = \hat{\beta} \circ f_*$. Since any continuous map $f : S^1 \to S^1$ has to be a loop, it follows from 7(a) that $f_* = g_*$, and consequently $deg(f) = deg(g)$.

   (b) To prove (b), we show first that $deg(f \circ g) = deg(f) \cdot deg(g)$. But this follows directly from the fact that $(f \circ g)_* = f_* \circ g_*$. Also, by definition

$deg(id) = 1$, as it induces the identity homomorphism. Therefore, if f is a homeomorphism, then $deg(f) \cdot \deg(f^{-1}) = 1$. As $deg(f)$ is an integer, we have that $deg(f) = \pm 1$. Finally, since a is a homeormorphism that is non-homotopic to f, we conclude that $deg(a) = -1$.